ONLINE APPENDIX FOR
Understanding Persistent ZLB: Theory and Assessment[⋆]

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A. Proofs for Section I

This section describes how we obtain the equations of the model of Section I, as well as proofs Propositions 1, 2 and 3.

A.1. Stationary Equilibrium

The stationary equilibrium of the baseline model is given by the following system of four equations in four stationary endogenous variables $\{\tilde{C}_t, \tilde{Y}_t, \Pi_t, R_t\}$ for a given exogenous sequence of variables $\{G_{z,t}, \nu_t, g_t, \delta_t\}$

$$1 = \beta \mathbb{E}_t \left[\left(\frac{\tilde{C}_{t+1}}{\tilde{C}_t} \right)^{-1} \frac{R_t}{G_{z,t+1} \Pi_{t+1}} \right] + \delta_t \tilde{C}_t \tag{A.1}$$

$$(1-\nu_t) - \omega h_t^{1/\eta} \tilde{C}_t + \nu_t \Phi' (\Pi_t) \Pi_t = \nu_t \beta \mathbb{E}_t \left[\left(\frac{\tilde{C}_{t+1}}{\tilde{C}_t} \right)^{-1} \Pi_{t+1} \Phi' (\Pi_{t+1}) \frac{\tilde{Y}_{t+1}}{\tilde{Y}_t} \right] \tag{A.2}$$

$$R_t = \max \left\{ 1, \tilde{R}_t \right\} \tag{A.3}$$

$$\tilde{C}_t + \tilde{G}_t = \tilde{Y}_t. \tag{A.4}$$

where $\tilde{C}_t \equiv \frac{C_t}{Z_t}$, $\tilde{Y}_t \equiv \frac{Y_t}{Z_t}$, $\tilde{G}_t \equiv \frac{G_t}{Z_t}$ are stationary variables and, $G_t = \left(1 - \frac{1}{g_t}\right) Y_t$. We choose $\omega = (1-\nu) g$ to normalize the full employment level of normalized output to $\tilde{Y} = 1$.

A.2. Approximate equilibrium around the Permanent Liquidity Trap

When the economy is at a permanent liquidity trap, we have R = 1. We denote by $\bar{x}$ the steady state values corresponding to the liquidity trap steady state. Variables with star denote the corresponding full-employment steady state value. Variables with hats and time-subscripts are log-deviations from the respective stationary steady state values.

$$\begin{aligned}
\hat{c}_t &= \frac{\beta}{(\beta + \bar{\pi} G_z \delta \bar{c})} \hat{c}_{t+1} + \frac{\beta}{(R\beta + \bar{\pi} G_z \delta \bar{c})} \left(\hat{\pi}_{t+1} + \hat{G}_{z,t+1} \right) \\
\hat{\pi}_t &= \beta \frac{(\bar{\pi} - \pi_*)}{2\bar{\pi} - \pi_*} \left[- (\hat{c}_{t+1} - \hat{c}_t) + \hat{y}_{t+1} - \hat{y}_t \right] + \beta \hat{\pi}_{t+1} + \left(\frac{1 - (1-\beta)\phi\bar{\pi}(\bar{\pi} - \pi_*)}{\phi\bar{\pi}(2\bar{\pi} - \pi_*)} \right) \hat{\nu}_t \\
&\quad + \left(\frac{(1-\nu) g_* \bar{c} \bar{y}^{1/\eta}}{\nu \phi \bar{\pi} (2\bar{\pi} - \pi_*)} \right) \left(\hat{c}_t + \frac{1}{\eta} \hat{y}_t \right) \\
\hat{c}_t &= \hat{y}_t - \hat{g}_t
\end{aligned}$$

Collecting terms and replacing the log-linearized resource constraint we have:

$$\hat{y}_t = \bar{\mathcal{D}}\mathbb{E}_t(\hat{y}_{t+1} - \hat{g}_{t+1}) + \bar{\mathcal{D}}\mathbb{E}_t\left(\hat{\pi}_{t+1} + \hat{G}_{z,t+1}\right) + \hat{g}_t \tag{A.5}$$
$$\hat{\pi}_t = \bar{\kappa}\left(\frac{\eta+1}{\eta}\hat{y}_t - \hat{g}_t\right) + \bar{\lambda}\hat{\nu}_t + \bar{\varphi}\mathbb{E}_t\Delta\hat{g}_{t+1} + \beta\mathbb{E}_t\hat{\pi}_{t+1}$$

Where $\bar{\mathcal{D}} = \frac{\beta}{(\beta+\bar{\pi}G_z\delta\bar{c})}$, $\bar{\lambda} = \left(\frac{1-(1-\beta)\phi\bar{\pi}(\bar{\pi}-\pi_*)}{\phi\bar{\pi}(2\bar{\pi}-\pi_*)}\right)$, $\bar{\kappa} = \left(\frac{(1-\nu)g_*\bar{c}\bar{y}^{1+1/\eta}}{\nu\phi\bar{\pi}(2\bar{\pi}-\pi_*)}\right)$, and $\bar{\varphi} = \beta\frac{(\bar{\pi}-\pi_*)}{2\bar{\pi}-\pi_*}$.

We obtain equation 5 from log-linearizing the consumption Euler equation 1. It resembles the dynamic IS relationship of the standard New Keynesian model but modified by the discount coefficient $\bar{\mathcal{D}}$. Since $\delta > 0$, the discounting coefficient $\bar{\mathcal{D}} < 1$. Discounting dampens the consumption response to changes in the ex-ante real interest rate. An increase in the preference for bonds, lower steady-state inflation, and lower long-run growth rate increase the discounting in the Euler equation conditional on $\delta > 0$. We introduce shocks to growth rate of technology, $G_{z,t}$, to replicate movements in the real interest rate observed in Japan.

Equation 6 is the forward-looking Phillips curve that depends on expected inflation and marginal costs $((1/\eta + 1)\,\hat{y}_t - \hat{g}_t)$, the growth in government expenditure $(\hat{g}_{t+1} - \hat{g}_t)$ and the price-markup shock $\hat{\upsilon}_t$. The growth in government expenditure appears in this equation because of we log-linearized the equation away from the targeted-inflation steady state.

Equation 7 is the resource constraint of the economy that specifies a time-varying wedge between consumption and output, corresponding to exogenous shocks in government spending. Equation 8 indicates that the economy operates under an interest rate peg. We can derive this equation from any policy rule in which the central bank faces an effective lower bound constraint.

A.3. Proof of Proposition 1

Without shocks, the system of equations around a permanent liquidity trap can be rewritten as:

$$\hat{y}_t = \bar{\mathcal{D}}\mathbb{E}_t(\hat{y}_{t+1} + \hat{\pi}_{t+1})$$

$$\hat{\pi}_t = \beta\mathbb{E}_t\hat{\pi}_{t+1} + \tilde{\kappa}\hat{y}_t$$

Where $\tilde{\kappa} = \frac{\eta+1}{\eta}\bar{\kappa}$, $\bar{\mathcal{D}} = \frac{\beta}{\beta+\bar{\pi}G_z\delta\bar{c}}$.

In matrix form, we can write the system as:

$$\begin{bmatrix} \bar{\mathcal{D}} & \bar{\mathcal{D}} \\ 0 & \beta \end{bmatrix}\begin{bmatrix} \hat{y}_{t+1} \\ \hat{\pi}_{t+1} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -\tilde{\kappa} & 1 \end{bmatrix}\begin{bmatrix} \hat{y}_t \\ \hat{\pi}_t \end{bmatrix}$$

Inverting the matrix on the left hand side, we get:

$$\begin{bmatrix} \hat{y}_{t+1} \\ \hat{\pi}_{t+1} \end{bmatrix} = \begin{bmatrix} (\phi + \tilde{\kappa}\rho) & -\rho \\ -\tilde{\kappa}\rho & \rho \end{bmatrix}\begin{bmatrix} \hat{y}_t \\ \hat{\pi}_t \end{bmatrix}$$

where $\rho \equiv 1/\beta$, and $\phi \equiv 1/\bar{\mathcal{D}}$. Define $M \equiv \begin{bmatrix} (\phi + \tilde{\kappa}\rho) & -\rho \\ -\tilde{\kappa}\rho & \rho \end{bmatrix}$. Then, we can derive the following properties of the matrix M:

$$det(M) = \phi\rho, \quad tr(M) = \phi + (1+\tilde{\kappa})\rho$$

Proposition C1 in (Woodford, 2003, pp 670) provides the necessary and sufficient conditions for determinacy for a system of 2 equations. A 2 × 2 matrix M with positive determinant has both eigenvalues outside the unit circle if and only if

$$\det M > 1, \quad \det M - \text{tr } M > -1, \quad \det M + \text{tr } M > -1$$

Under our sign restrictions on parameters and the assumption that $\beta < 1$, first and third inequalities necessarily hold. It follows then that both eigenvalues are outside the unit circle if and only if $\phi > \frac{1-\rho(1+\tilde{\kappa})}{1-\rho} = 1 - \frac{\rho\tilde{\kappa}}{1-\rho}$ for determinacy. This implies $1/\bar{\mathcal{D}} > \frac{\frac{\beta}{\beta} - \frac{1}{\beta}(1+\tilde{\kappa})}{\frac{\beta-1}{\beta}} = \frac{(\beta-1)-\tilde{\kappa}}{\beta-1} = 1 + \frac{\tilde{\kappa}}{1-\beta}$. We can rewrite this inequality to obtain $\frac{1-\beta}{1-\beta+\tilde{\kappa}} > \bar{\mathcal{D}}$, which yields the restriction in the proposition.

A.4. Proof of Proposition 2

This section derives the unique solution for secular stagnation, given local determinacy.

Guess that $\hat{y}_t = a_1\hat{G}_{z,t} + a_2\hat{g}_t + a_3\hat{\nu}_t$ and $\hat{\pi}_t = b_1\hat{G}_{z,t} + b_2\hat{g}_t + b_3\hat{\nu}_t$ and solve for the unknown $a's$ and $b's$. Replacing the guess into (A.5) and dropping time subscripts, we obtain:

$$\begin{aligned} a_1\hat{G}_{z,t} + a_2\hat{g}_t + a_3\hat{\nu}_t &= \bar{\mathcal{D}}\left(a_1\rho_z\hat{G}_{z,t} + a_2\rho_g\hat{g}_t + a_3\rho_v\hat{\nu}_t - \rho_g g + b_1\rho_z\hat{G}_{z,t} + b_2\rho_g\hat{g}_t + b_3\rho_v\hat{\nu}_t + \rho_z\hat{G}_{z,t}\right) + \hat{g}_t \\ &= \bar{\mathcal{D}}(a_1\rho_z + b_1\rho_z + \rho_z)\hat{G}_{z,t} + (\bar{\mathcal{D}}a_2\rho_g - \bar{\mathcal{D}}\rho_g + \bar{\mathcal{D}}b_2\rho_g + 1)\hat{g}_t + \bar{\mathcal{D}}\left(b_3\rho_v + a_3\rho_v\right)\hat{\nu}_t \end{aligned}$$

$$\begin{aligned} b_1\hat{G}_{z,t} + b_2\hat{g}_t + b_3\hat{\nu}_t &= \beta\left(b_1\rho_z\hat{G}_{z,t} + b_2\rho_g\hat{g}_t + b_3\rho_v\hat{\nu}_t\right) + \bar{\varphi}\left(\rho_g - 1\right)\hat{g}_t + \bar{\lambda}\hat{\nu}_t \\ &+ \bar{\kappa}(\frac{1}{\eta} + 1)\left(a_1\hat{G}_{z,t} + a_2\hat{g}_t + a_3\hat{\nu}_t\right) - \bar{\kappa}\hat{g}_t \\ &= \left(\beta b_1\rho_z + \bar{\kappa}(\frac{1}{\eta} + 1)a_1\right)\hat{G}_{z,t} + \left(\beta b_2\rho_g + \bar{\varphi}\left(\rho_g - 1\right) + \bar{\kappa}(\frac{1}{\eta} + 1)a_2 - \bar{\kappa}\right)\hat{g}_t \\ &+ \left(\beta b_3\rho_v + \bar{\kappa}(\frac{1}{\eta} + 1)a_3 + \bar{\lambda}\right)\hat{\nu}_t \end{aligned}$$

Comparing terms we can write the following system of equations:

$$\begin{bmatrix} 0 \\ -\bar{\varphi}\left(\rho_g - 1\right) + \bar{\kappa} \\ -\bar{\lambda} \\ -\bar{\mathcal{D}}\rho_z \\ \bar{\mathcal{D}}\rho_g - 1 \\ 0 \end{bmatrix} = \begin{bmatrix} (\beta\rho_z - 1) & 0 & 0 & \bar{\kappa}(\frac{1}{\eta}+1) & 0 & 0 \\ 0 & (\beta\rho_g - 1) & 0 & 0 & \bar{\kappa}(\frac{1}{\eta}+1) & 0 \\ 0 & 0 & (\beta\rho_v - 1) & 0 & 0 & \bar{\kappa}(\frac{1}{\eta}+1) \\ \bar{\mathcal{D}}\rho_z & 0 & 0 & \left(\bar{\mathcal{D}}\rho_z - 1\right) & 0 & 0 \\ 0 & \bar{\mathcal{D}}\rho_g & 0 & 0 & \left(\bar{\mathcal{D}}\rho_g - 1\right) & 0 \\ 0 & 0 & \bar{\mathcal{D}}\rho_v & 0 & 0 & \left(\bar{\mathcal{D}}\rho_v - 1\right) \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix}$$

The solution is:

$$b_1 = \frac{-\bar{\mathcal{D}}\rho_Z\bar{\kappa}\left(\frac{1}{\eta}+1\right)}{\bar{\mathcal{D}}\rho_z\left[\bar{\kappa}\left(\frac{1}{\eta}+1\right) + (1-\beta\rho_z)\right] - (1-\beta\rho_z)}; \qquad a_1 = \frac{(1-\beta\rho_z)b_1}{\bar{\kappa}\left(\frac{1}{\eta}+1\right)} \tag{A.6}$$

$$b_2 = \frac{(1-\bar{\mathcal{D}}\rho_g)\left[\bar{\varphi}(1-\rho_g) - \bar{\kappa}\frac{1}{\eta}\right]}{\bar{\mathcal{D}}\rho_g\left[\bar{\kappa}\left(\frac{1}{\eta}+1\right)\right] - (1-\beta\rho_g)(1-\bar{\mathcal{D}}\rho_g)}; \qquad a_2 = \frac{\bar{\kappa} + (1-\beta\rho_g)b_2 + \bar{\varphi}(1-\rho_g)}{\bar{\kappa}\left(\frac{1}{\eta}+1\right)} \tag{A.7}$$

$$b_3 = \frac{-\bar{\lambda}(1-\bar{\mathcal{D}})\rho_v}{\bar{\mathcal{D}}\rho_v\left[\bar{\kappa}\left(\frac{1}{\eta}+1\right) + (1-\beta\rho_v)\right] - (1-\beta\rho_v)}; \qquad a_3 = \frac{\bar{\mathcal{D}}\rho_v b_3}{1-\bar{\mathcal{D}}\rho_v} \tag{A.8}$$

A.5. Proof of Proposition 3

Proof. To construct the proof for correlation, first we compute various moments. Note that:

$$\mathbb{E}[\hat{y}_t] = \mathbb{E}[\hat{\pi}_t] = \mathbb{E}[\hat{G}_{z,t}] = \mathbb{E}[\hat{g}_t] = \mathbb{E}[\hat{\nu}_t] = 0.$$

$$\mathbb{E}[\hat{G}_{z,t}^2] = \frac{1}{1-\rho_z^2}\sigma_z^2; \quad \mathbb{E}[\hat{g}_t^2] = \frac{1}{1-\rho_g^2}\sigma_g^2; \quad \mathbb{E}[\hat{\nu}_t^2] = \frac{1}{1-\rho_\nu^2}\sigma_\nu^2$$

From the solution $\{a_i, b_i\}$ $\forall i \in [1,2,3]$ derived in Proposition 2, we can write output growth and inflation as follows:

$$\hat{y}_t - \hat{y}_{t-1} = a_1\epsilon_{z,t} + a_2\epsilon_{g,t} + a_3\epsilon_{\nu,t} - (1-\rho_z)a_1\hat{z}_{t-1} - (1-\rho_g)a_2\hat{g}_{t-1} - (1-\rho_\nu)a_3\hat{\nu}_{t-1}$$

$$\hat{\pi}_t = b_1\epsilon_{z,t} + b_2\epsilon_{g,t} + b_3\epsilon_{\nu,t} + \rho_z b_1\hat{z}_{t-1} + \rho_g b_2\hat{g}_{t-1} + \rho_\nu b_3\hat{\nu}_{t-1}$$

Correlation between output growth and inflation is then given by:

$$\rho_{dy_t,\hat{\pi}_t} = \frac{\mathbb{E}\left[\left(\hat{y}_t - \hat{y}_{t-1}\right)\pi_t\right]}{\sqrt{Var\left(\hat{y}_t - \hat{y}_{t-1}\right)Var(\pi_t)}}$$

The correlation is positive if and only if the numerator is positive. Evaluating the numerator, we get:

$$\begin{aligned}\mathbb{E}\left[\left(\hat{y}_t - \hat{y}_{t-1}\right)\hat{\pi}_t\right] = {} & a_1b_1\sigma_z^2 + a_2b_2\sigma_g^2 + a_3b_3\sigma_\nu^2 \\ & - \left((1-\rho_z)\rho_z a_1b_1\mathbb{E}[\hat{z}_{t-1}^2] + (1-\rho_g)\rho_g a_2b_2\mathbb{E}[\hat{g}_{t-1}^2] + (1-\rho_\nu)\rho_\nu a_3b_3\mathbb{E}[\hat{\nu}_t^2]\right)\end{aligned}$$

This can be simplified to:

$$\mathbb{E}\left[\left(\hat{y}_t - \hat{y}_{t-1}\right)\hat{\pi}_t\right] = \frac{a_1 b_1}{1+\rho_z}\sigma_z^2 + \frac{a_2 b_2}{1+\rho_g}\sigma_g^2 + \frac{a_3 b_3}{1+\rho_\nu}\sigma_\nu^2$$

From the solution $\{a_i, b_i\}$ $\forall i \in [1,2,3]$ derived in Proposition 2, note that the products $a_1 b_1$ and $a_3 b_3$ are non-negative. Therefore, conditional of technology growth rate shocks and price-markups shocks, inflation and output growth are (weakly) positively correlated. Positive correlation between inflation and output growth also obtains under government spending shocks if $\bar{\kappa} < \frac{\bar{\pi} z \bar{c} \delta(1-\beta)}{\beta}$.

We can use the matrix equations to alternately rewrite output and inflation IRF to govt spending shock as follows.

$$a_2 = \frac{1 + \bar{\mathcal{D}}\rho_g(b_2 - 1)}{1 - \bar{\mathcal{D}}\rho_g}$$

Consequently, $a_2 > 0$ whenever $b_2 > 0$.

When $b_2 < 0$, a condition that guarantees that $a_2 < 0$ is $b_2 < -\frac{\bar{\kappa}}{(1-\beta)}$ (From A.7 and the fact that $\bar{\varphi} < 0$). Rewrite this condition, and substitute in the values of parameters to obtain the requirement that $\bar{\kappa} < \frac{\bar{\pi} z \bar{c} \delta(1-\beta)}{\beta}$ is sufficient for positive correlation between inflation and output growth.

This latter condition is implied by the local determinacy requirement discussed in Proposition 1. We then assumed that:

$$\frac{\tilde{\pi} G_z \delta \tilde{c}}{\beta} > \frac{1+\eta}{\eta(1-\beta)}\bar{\kappa}$$

which implies $\bar{\kappa} < \frac{\bar{\pi} z \bar{c} \delta(1-\beta)}{\beta}$.

□

A.6. Proof of Proposition 4

Before we construct the proof for Proposition 4, we describe the non-linear equations and approximate equilibrium.

Bilbiie (2021) provides a micro foundation for the static Phillips curve assumed in Section I. Price adjustment costs are postulated in the tradition of "external" habits. Adjustment costs are such that firms consider yesterday's market average price index instead of their own individual last-period price. That is, we assume that the adjustment costs take the following form:

$$\Phi_s \equiv \frac{\phi}{2}\left(\frac{P_t(j)}{P_{t-1}} - \Pi^*\right)^2 Y_t$$

The stationary equilibrium of the model with static Phillips curve is given by the following system of four equations in four stationary endogenous variables $\{\tilde{C}_t, \tilde{Y}_t, \Pi_t, R_t\}$ for a given exogenous sequence of variables $\{G_{z,t}, \nu_t, g_t, \delta_t\}$

$$1 = \beta \mathbb{E}_t \left[\left(\frac{\tilde{C}_{t+1}}{\tilde{C}_t} \right)^{-1} \frac{R_t}{G_{z,t+1}\Pi_{t+1}} \right] + \delta_t \tilde{C}_t \tag{A.9}$$

$$(1-\nu_t) - \omega h_t^{1/\eta} \tilde{C}_t + \nu_t \Phi'(\Pi_t) \Pi_t = 0 \tag{A.10}$$

$$R_t = \max \left\{ 1, \tilde{R}_t \right\} \tag{A.11}$$

$$\tilde{C}_t + \tilde{G}_t = \tilde{Y}_t. \tag{A.12}$$

where $\tilde{C}_t \equiv \frac{C_t}{Z_t}$, $\tilde{Y}_t \equiv \frac{Y_t}{Z_t}$, $\tilde{G}_t \equiv \frac{G_t}{Z_t}$ are stationary variables and, $G_t = \left(1 - \frac{1}{g_t}\right) Y_t$. We choose $\omega = (1-\nu)g_*$ to normalize the full employment level of normalized output to $\tilde{Y} = 1$.

When the economy is in a permanent liquidity trap, we have R = 1. We denote by $\bar{x}_s$ the steady state values corresponding to the liquidity trap steady state with a static Phillips curve. Variables with a star denote the corresponding steady-state value for full employment. Variables with hats and time subscripts are log deviations from the stationary steady-state values. π_t is the log deviation of gross inflation from the steady state.

$$\hat{c}_t = \frac{\beta}{(\beta + \bar{\pi}_s G_z \delta \bar{c}_s)} \hat{c}_{t+1} + \frac{\beta}{(R\beta + \bar{\pi}_s G_z \delta \bar{c}_s)} \left(\hat{\pi}_{t+1} + \hat{G}_{z,t+1} \right)$$

$$\hat{\pi}_t = \left(\frac{1 - \phi \bar{\pi}_s (\bar{\pi}_s - \pi_*)}{\phi \bar{\pi}_s (2\bar{\pi}_s - \pi_*)} \right) \hat{\nu}_t + \left(\frac{(1-\nu) g_* \bar{c}_s \bar{y}_s^{1/\eta}}{\nu \phi \bar{\pi}_s (2\bar{\pi}_s - \pi_*)} \right) \left(\hat{c}_t + \frac{1}{\eta} \hat{y}_t \right)$$

$$\hat{c}_t = \hat{y}_t - \hat{g}_t$$

Collecting terms and replacing the log-linearized resource constraint, we have:

$$\hat{y}_t = \bar{\mathcal{D}}(\hat{y}_{t+1} - \hat{g}_{t+1}) + \bar{\mathcal{D}} \left(\hat{\pi}_{t+1} + \hat{G}_{z,t+1} \right) + \hat{g}_t \tag{A.13}$$

$$\hat{\pi}_t = \bar{\kappa} \left(\frac{\eta + 1}{\eta} \hat{y}_t - \hat{g}_t \right) + \bar{\lambda} \hat{\nu}_t$$

Where $\bar{\mathcal{D}} = \frac{\beta}{(\beta + \bar{\pi}_s G_z \delta \bar{c}_s)}$, $\bar{\lambda} = \left(\frac{1 - \phi \bar{\pi}_s (\bar{\pi}_s - \pi_*)}{\phi \bar{\pi}_s (2\bar{\pi}_s - \pi_*)} \right)$, and $\bar{\kappa} = \left(\frac{(1-\nu) g_* \bar{c}_s \bar{y}_s^{1+1/\eta}}{\nu \phi \bar{\pi}_s (2\bar{\pi}_s - \pi_*)} \right)$. Shutting down the government spending shocks and the TFP growth rate shocks, we obtain

$$\hat{y}_t = \bar{\mathcal{D}} \left(\hat{y}_{t+1} + \hat{\pi}_{t+1} \right) \tag{A.14}$$

$$\hat{\pi}_t = \tilde{\kappa} \hat{y}_t + \bar{\lambda} \hat{\nu}_t$$

where $\tilde{\kappa} = \bar{\kappa} \left(\frac{\eta + 1}{\eta} \right)$. Substituting the Phillips curve into the Euler equation, we arrive at the system of log-linearized equations presented in the main text.

A.6.1 Baseline Case: Sunspot on Inflation Forecast Error

We define the one-step ahead forecast error associated with the expectational variable $\hat{\pi}_t$, as:

$$\zeta_t \equiv \hat{\pi}_t - \mathbb{E}_{t-1}\hat{\pi}_t \tag{A.15}$$

Because we are analyzing the system around a locally indeterminate steady state, $\Lambda > 1$. We combine this equation with equation (A.15), to get the solutions of the following form:

$$\hat{\pi}_t = \Lambda^{-1}\ \hat{\pi}_{t-1} - \Lambda^{-1}\ \bar{\lambda}\hat{\nu}_{t-1} + \zeta_t$$

$$\hat{y}_t = \tilde{\kappa}^{-1}\hat{\pi}_t - \bar{\lambda}\tilde{\kappa}^{-1}\hat{\nu}_t$$

In a stationary solution, the unconditional means of $\hat{y}_t$ and $\hat{\pi}_t$ are zero. The expression for the variance of $\hat{\pi}_t$ is:

$$\sigma_\pi^2 = \frac{\bar{\lambda}^2}{\Lambda^2 - 1}\sigma_\nu^2 + \frac{\Lambda^2}{\Lambda^2 - 1}\sigma_\zeta^2 - \frac{\lambda}{\Lambda^2 - 1}\rho_{\nu,\zeta}\sigma_\nu\sigma_\zeta \tag{A.16}$$

where $\sigma_\zeta^2 \equiv E\zeta_t^2$, and $\rho_{\nu,\zeta} \equiv \frac{E(\nu_t\zeta_t)}{\sigma_\nu\sigma_\zeta}$. Furthermore,

$$\mathbb{E}\left[\left(\hat{y}_t - \hat{y}_{t-1}\right)\hat{\pi}_t\right] = \tilde{\kappa}^{-1}\left[\frac{-\bar{\lambda}^2\sigma_\nu^2 + \Lambda\sigma_\zeta^2 - (\Lambda - 1)\bar{\lambda}\rho_{\nu,\zeta}\sigma_\nu\sigma_\zeta}{\Lambda + 1}\right]$$

Thus, the correlation between inflation and output growth is negative if and only if:

$$1 > \rho_{\nu,\zeta} > \frac{\Lambda\sigma_\zeta^2 - \bar{\lambda}^2\sigma_\nu^2}{(\Lambda - 1)\bar{\lambda}\sigma_\nu\sigma_\zeta}$$

A.6.2 Extension: Sunspot on Output Forecast Error

We consider a variation of the baseline setup. We define the sunspot on the output forecast error instead of the inflation forecast error. We define the one-step ahead forecast error associated with output as:

$$\zeta_{y,t} \equiv \hat{y}_t - \mathbb{E}_{t-1}\hat{y}_t$$

As above, for local indeterminacy, it follows that $\Lambda \equiv \bar{\mathcal{D}}\left(1 + \tilde{\kappa}\right) > 1$. The solution to this system takes the following form:

$$\hat{y}_t = \Lambda^{-1}\hat{y}_{t-1} + \zeta_{y,t}$$
$$\hat{\pi}_t = \tilde{\kappa}\hat{y}_t + \bar{\lambda}\hat{\nu}_t$$

In a stationary solution, the unconditional means of $\hat{y}_t$ and $\hat{\pi}_t$ are zero. The expression for the variance of $\hat{y}_t$ is:

$$\sigma_y^2 = \frac{\sigma_\zeta^2}{1-\Lambda^{-2}} = \frac{\Lambda^2}{\Lambda^2-1}\sigma_\zeta^2 \tag{A.17}$$

where $\sigma_\zeta^2 \equiv E\zeta_{y,t}^2$ with slight abuse of notation.

In order to compute correlation between inflation and output growth,

$$\begin{aligned}\mathbb{E}\left[\left(\hat{y}_t - \hat{y}_{t-1}\right)\hat{\pi}_t\right] = \mathbb{E}\left(\left[(\Lambda^{-1}-1)\hat{y}_{t-1} + \zeta_{y,t}\right]\left[\Lambda^{-1}\tilde{\kappa}\hat{y}_{t-1} + \tilde{\kappa}\zeta_{y,t} + \bar{\lambda}\hat{\nu}_t\right]\right) &= \frac{\tilde{\kappa}\sigma_\zeta^2}{1+\Lambda^{-1}} + \bar{\lambda}\mathbb{E}\left[\nu_t\zeta_t\right] \\ &= \frac{\tilde{\kappa}\sigma_\zeta^2}{1+\Lambda^{-1}} + \bar{\lambda}\rho_{\nu,\zeta}\sigma_\nu\sigma_\zeta\end{aligned}$$

where $\rho_{\nu,\zeta} \equiv \frac{\mathbb{E}(\nu_t\zeta_t)}{\sigma_\nu\sigma_\zeta}$. The correlation of inflation with output growth is negative if and only if:

$$-1 \leq \rho_{\nu,\zeta} < -\frac{\tilde{\kappa}\sigma_\zeta}{\left(1+\Lambda^{-1}\right)\bar{\lambda}\sigma_\nu}$$

B. Extensions to Stylized Model

B.1. Description of secular stagnation without rational expectations

We provide the equilibrium conditions for variants of the secular stagnation model with non-rational agents or heterogenous agents.

B.1.1 Bordalo, Gennaioli and Shleifer (2018) diagnostic agents

The non-linear equilibrium conditions are similar to those described in Appendix A.1, with the main exception that we replace the rational expectations operator with the diagnostic expectations operator $\mathbb{E}_t^\theta$.

The stationary equilibrium of the baseline model with diagnostic expectations is given by the following system of four equations in four stationary endogenous variables $\{\tilde{C}_t, \tilde{Y}_t, \Pi_t, R_t\}$ for a given exogenous sequence of variables $\{G_{z,t}, \nu_t, \tilde{G}_t, \delta_t\}$

$$1 = \beta R_t G_{z,t}\Pi_t \mathbb{E}_t^\theta\left[\left(\frac{\tilde{C}_{t+1}}{\tilde{C}_t}\right)^{-1}\frac{1}{G_{z,t}G_{z,t+1}\Pi_t\Pi_{t+1}}\right] + \delta_t\tilde{C}_t \tag{B.1}$$

$$(1-\nu_t) - \omega h_t^{1/\eta}\tilde{C}_t + \nu_t\Phi'\left(\Pi_t\right)\Pi_t = \nu_t\beta\mathbb{E}_t^\theta\left[\left(\frac{\tilde{C}_{t+1}}{\tilde{C}_t}\right)^{-1}\Pi_{t+1}\Phi'\left(\Pi_{t+1}\right)\frac{\tilde{Y}_{t+1}}{\tilde{Y}_t}\right] \tag{B.2}$$

$$R_t = \max\left\{1, \tilde{R}_t\right\} \tag{B.3}$$

$$\tilde{C}_t + \tilde{G}_t = \tilde{Y}_t. \tag{B.4}$$

where $\tilde{C}_t \equiv \frac{C_t}{Z_t}$, $\tilde{Y}_t \equiv \frac{Y_t}{Z_t}$, and $\tilde{G}_t \equiv \frac{G_t}{Z_t}$ are stationarized variables. We choose $\omega = (1-\nu)g_*$ to normalize the full employment level of normalized output to $\tilde{Y} = 1$.

Following L'Huillier, Singh and Yoo (2023), we employ diagnostic expectations such that the steady state is unchanged relative to the model with rational expectations. In a linearized general equilibrium model, the diagnostic expectations operator on a future variable is defined as:

$$\mathbb{E}_t^\theta[X_{t+1}] = (1+\theta)\mathbb{E}_t[X_{t+1}] - \theta\mathbb{E}_{t-1}[X_{t+1}]; \quad \theta \geq 0$$

The log-linearized system then is given by System B.5:

$$\hat{y}_t = \bar{\mathcal{D}}\mathbb{E}_t^\theta\left(\hat{y}_{t+1} - \hat{g}_{t+1} + \hat{\pi}_{t+1} + \hat{G}_{z,t+1}\right) + \hat{g}_t + \theta\left(\hat{\pi}_t - \mathbb{E}_{t-1}\hat{\pi}_t + \hat{G}_{z,t} - \mathbb{E}_{t-1}\hat{G}_{z,t}\right) \tag{B.5}$$
$$\hat{\pi}_t = \bar{\kappa}\left(\frac{\eta+1}{\eta}\hat{y}_t - \hat{g}_t\right) + \bar{\lambda}\hat{\nu}_t + \bar{\varphi}(\mathbb{E}_t^\theta\hat{g}_{t+1} - \hat{g}_t) + \beta\mathbb{E}_t^\theta\hat{\pi}_{t+1}$$

Where $\bar{\mathcal{D}} = \frac{\beta}{(\beta+\bar{\pi}G_z\delta\bar{c})}$, $\bar{\lambda} = \left(\frac{1-(1-\beta)\phi\bar{\pi}(\bar{\pi}-\pi_*)}{\phi\bar{\pi}(2\bar{\pi}-\pi_*)}\right)$, $\bar{\kappa} = \left(\frac{(1-\nu)g_*\bar{c}\bar{y}^{1+1/\eta}}{\nu\phi\bar{\pi}(2\bar{\pi}-\pi_*)}\right)$, and $\bar{\varphi} = \beta\frac{(\bar{\pi}-\pi_*)}{2\bar{\pi}-\pi_*}$.

B.1.2 Gabaix (2020) behavioral agents

We replace the rational expectations operator $\mathbb{E}_t$ in the secular stagnation equilibrium with Gabaix's bounded rationality $\mathbb{E}_t^{BR}$ operator. In the steady state, $\mathbb{E}^{BR} = \mathbb{E}$. In making a forecast for variable X_{t+k} which is $k > 0$ periods in the future, the Gabaix's bounded rationality operator is linked to the rational expectations operator in the following manner:

$$\mathbb{E}_t^{BR}[X_{t+k}] = \bar{m}^k\mathbb{E}_t[X_{t+k}]$$

where $\bar{m} \in [0,1]$.

With the exception of this change, the equilibrium conditions are same as described in System A.5.

B.1.3 Bilbiie (2021) heterogenous agents with behavioral expectations

We consider the case of zero steady state inequality, which is the baseline case in Bilbiie (2021). The focus is thus only on the cyclical inequality. This version has the advantage that we can maintain same steady state as our baseline model, introduced in Section I.

The log-linearized Euler-equation would however be different:

$$\hat{y}_t = \mathcal{D}^T\Gamma(\hat{y}_{t+1} - \hat{g}_{t+1} + \hat{G}_{z,t+1}) + \mathcal{D}^T\frac{1-\lambda}{1-\lambda\chi}\hat{\pi}_{t+1} + \mathcal{D}^T\zeta\left[\frac{\lambda(\chi-1)}{1-\lambda\chi}(g_t - E_tg_{t+1}) + (\Gamma-1)E_tg_{t+1}\right] + g_t$$

where $\mathcal{D}^T \equiv \left(\frac{\pi\gamma\delta c}{\beta}\frac{1-\lambda\chi}{1-\lambda} + 1\right)^{-1}$, $\Gamma \equiv 1 + (\chi-1)\frac{1-s}{1-\lambda\chi}$, $\chi \equiv (1+\eta^{-1})\left(1 - \frac{\tau^D}{\lambda}\right)$, $\zeta \equiv 1/(1+\eta)$, and $\lambda = \frac{1-s}{2-s-h}$. As in Bilbiie, λ is the unconditional mass of the hand-to-mouth agents, s and h are the probabilities that a saver and hand-to-mouth agent stay in their respective states, η is the Frisch elasticity of labor supply, τ^D is the rate at which profits are taxed, χ is a measure of cyclicality of income inequality. We follow Bilbiie (2021) to set $\chi = 1.48$, $\lambda = 0.37$, and $1 - s = 0.04$. Other parameters are same as in our baseline calibration. The Phillips curve is unchanged relative to the baseline:

$$\hat{\pi}_t = \bar{\kappa}\left(\frac{\eta+1}{\eta}\hat{y}_t - \hat{g}_t\right) + \bar{\lambda}\hat{\nu}_t + \bar{\varphi}(\mathbb{E}_t\hat{g}_{t+1} - \hat{g}_t) + \beta\mathbb{E}_t\hat{\pi}_{t+1}$$

In addition, the behavioral THANK model augments $\bar{m}$ in front of the rational expectations operator. We maintain Gabaix (2020) calibration of $\bar{m} = 0.85$, suggested also by Pfäuti and Seyrich (2022).

C. DSGE ESTIMATION

C.1. Data sources

We collect quarterly nominal GDP (GDP), nominal Consumption (C), nominal Investment (I), the GDP Deflator (DEF), total population (POP), hours worked (H), Earnings (W) from 1990Q1 to 2019Q4. We obtained all data from the Cabinet Office of Japan and the Ministry of Health Labor and Welfare. All data was obtained through Haver Analytics. The associated mnemonics are given in parenthesis.

GDP: is nominal gross domestic product (N9DP2) in billions of yen (seasonally adjusted at annual rate).
I: is the sum of nominal gross private domestic fixed investment (NEDFP2) and the change in private sector inventories (NEDSP2). Both series in billions of Yen, and seasonally adjusted.
G: is nominal government final consumption expenditure (N9GC2) in billions of Yen and is seasonally adjusted.
C: We collect nominal private final consumption expenditure (N9PC2), seasonally adjusted and in billions of yen.
DEF: is the gross domestic product deflator (JPSDEDP) seasonally adjusted and defined such that $2015 = 100$.
POP: is the total population of 15 years old and over (FL15), in 10,000 persons. Source: Ministry of Internal Affairs and Communications.
TH: corresponds to the total monthly hours worked per employee in companies with 30 or more employees—all industries (JPEWTH2). **H**: corresponds to monthly scheduled hours worked per employee in companies with 30 or more employees—all industries (MHA).
W: Corresponds to contractual earnings index (2020=100) for companies with 30 or more employees—all industries (JPEWR2).

C.1.1 Data transformations

Population and the call rate are converted from monthly to quarterly data by taking quarterly averages. We construct per-capita GDP, per-capita consumption, and investment, dividing the nominal quantities by population. We deflate per-capita variables using the GDP deflator. Lastly, we compute the quarter-on-quarter log difference of real per-capita GDP, real per-capita Consumption, and real per-capita investment and multiplied by 100. Inflation is defined as the quarter-on-quarter log difference of the GDP deflator and multiplied by 400 to convert it into annualized percentages.

We follow the same procedure detailed in Hirose (2020) to construct wages and hours worked per capita. We first construct average working days per month from the contractual hours series (H) and assume a constant eighth-hour workday. Then, we calculate the average hourly wage using total hours worked (TH) and contractual earnings (W). We compute log deviations of total daily hours worked and daily wages per hour with respect to their respective averages in our estimation sample.

For Figure 5a, we additionally use data on six-to-ten-year inflation expectations over the sample 1998Q1:2012Q4. This data was borrowed from the replication package of Aruoba, Cuba-Borda and Schorfheide (2018), who download it from the G7 Long-term Forecasts of Consensus Economics. This is a survey-based measure based on professional forecasters. Long-term forecasts are released in April and October of every year. To construct quarterly series they use linear interpolation. We follow their procedure to map the raw data to the model.

C.2. Measurement equations

Stylized model

To match the model to the data, we construct model implied output (Δy_t^o), consumption growth (Δc_t^o), as quarter-on-quarter percentages, and inflation measured in annualized percentages (π_t^o). We link the observed data series to the model counterparts through the following system of measurement equations:

$$\begin{aligned}
\Delta y_t^o &= 100\log(G_z) + 100\left(\hat{y}_t - \hat{y}_{t-1} + \hat{G}_{z,t}\right), \\
\Delta c_t^o &= 100\log(G_z) + 100\left(\hat{c}_t - \hat{c}_{t-1} + \hat{G}_{z,t}\right), \\
\pi_t^o &= 400\log\left(\bar{\pi}\right) + 400\hat{\pi}_t.
\end{aligned} \tag{C.1}$$

Medium scale model

In the medium scale model, in addition to output growth, consumption growth, and inflation, we construct model implied investment growth (Δi_t^o), real wage growth (Δw_t^o) and, hours worked (l_t^o). We augment measurement equations (C.1) with the following relations:

$$\begin{aligned}
\Delta \mathbb{I}_t^o &= 100\log(G_z) + 100\left(\hat{\mathbb{I}}_t - \hat{\mathbb{I}}_{t-1} + \hat{G}_{z,t}\right), \\
\Delta w_t^o &= 100\log(G_z) + 100\left(\hat{w}_t - \hat{w}_{t-1} + \hat{G}_{z,t}\right), \\
l_t^o &= 100\log(\bar{l}) + 100\hat{l}_t + \sigma_l \epsilon_t^l.
\end{aligned} \tag{C.2}$$

Where $\epsilon_t^l \sim N(0, \sigma_l^2)$ is the measurement error that helps the model fit high-frequency movements in the hours-worked series. We set the variance of the measurement error σ_l^2 to be 10% of the sample variance in l_t^o.

C.3. Prior distributions

Table C.1 lists the priors used to estimate the DSGE model of Section II, including information on the marginal prior distributions for the estimated parameters. Under the prior, we assume that all estimated parameters are distributed independently, which implies that the joint prior distribution can be computed from the product of the marginal distributions.

Table C.1: *Prior Distribution of DSGE parameters*

Parameters	Description	Distribution	P(1)	P(2)
ρ_z	Persistence tech. growth shock	$\mathcal{B}$	0.5	0.15
ρ_{ν_p}	Persistence price markup shock	$\mathcal{B}$	0.5	0.15
ρ_g	Persistence gov. spending shock	$\mathcal{B}$	0.5	0.15
ρ_μ	Persistence MEI shock	$\mathcal{B}$	0.5	0.15
ρ_{ν_w}	Persistence wage markup shock	$\mathcal{B}$	0.5	0.15
ρ_η	Persistence risk premium shock	$\mathcal{B}$	0.5	0.15
σ_z	Std dev. tech. growth shock	$\mathcal{IG}$	0.005	Inf
σ_{ν_p}	Std dev. price markup shock	$\mathcal{IG}$	0.005	Inf
σ_g	Std dev. gov. spending shock	$\mathcal{IG}$	0.005	Inf
σ_μ	Std dev. MEI shock	$\mathcal{IG}$	0.005	Inf
σ_{ν_w}	Std dev. wage markup shock	$\mathcal{IG}$	0.005	Inf
σ_η	Std dev. risk premium shock	$\mathcal{IG}$	0.005	Inf
σ_ζ	Std dev. sunspot shock	$\mathcal{IG}$	0.005	Inf
$\rho(\epsilon_z, \zeta)$	Corr. sun. and tech. growth shocks	$\mathcal{U}$	0	0.5774
$\rho(\epsilon_p, \zeta)$	Corr. sun. and price markup shocks	$\mathcal{U}$	0	0.5774
$\rho(\epsilon_g, \zeta)$	Corr. sun. and gov. spending shocks	$\mathcal{U}$	0	0.5774
$\rho(\epsilon_\mu, \zeta)$	Corr. sun. and MEI shocks	$\mathcal{U}$	0	0.5774
$\rho(\epsilon_w, \zeta)$	Corr. sun. and wage markup shocks	$\mathcal{U}$	0	0.5774
$\rho(\epsilon_\eta, \zeta)$	Corr. sun. and risk premium shocks	$\mathcal{U}$	0	0.5774

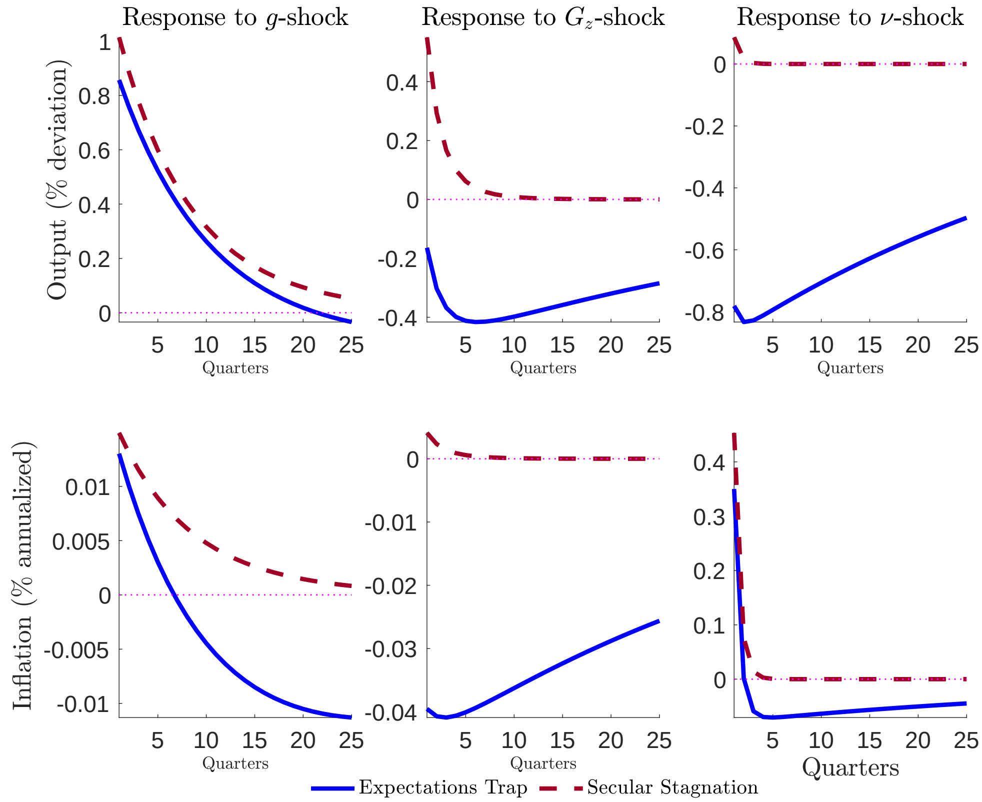
Notes: $\mathcal{G}$ is Gamma distribution; $\mathcal{B}$ is Beta distribution; $\mathcal{IG}$ is Inverse Gamma distribution; and $\mathcal{U}$ is Uniform distribution. P(1) and P(2) are mean and standard deviations for Beta, Gamma, and Uniform distributions.

C.4. Posterior sampler

We can solve the log-linearized system of equations of Section II using standard perturbation techniques. As a result, the likelihood function can be evaluated with the Kalman filter. We generate draws from the posterior distribution using the random walk Metropolis algorithm (RWM) described in An and Schorfheide (2007). We scale the covariance matrix of the proposal distribution in the RWM algorithm to obtain an acceptance rate between 30% and 50%. We generated 100,000 draws from the posterior distribution and discarded the first 50,000 draws.

C.5. Impulse Responses

Figure C.1: *Impulse Responses: Expectations Trap vs Secular Stagnation*



Notes: Impulse responses to one standard deviation shocks. All responses are computed at the posterior mean of the estimated parameters. The blue solid line corresponds to the expectations-driven traps model. The red dashed line corresponds to the secular stagnation model.

C.6. Additional Estimation Results

This section presents additional estimation results of our benchmark model. Section III.B.2 we extend the expectations-trap model with inflation expectations data (Infl. Exp. Data). In Section III.C we extend the secular stagnation model to allow for diagnostic expectations (DE), cognitive discounting (Gabaix), behavioral and hand-to-mouth agents (BTHANK).

Table C.2: *Posterior DSGE estimates*

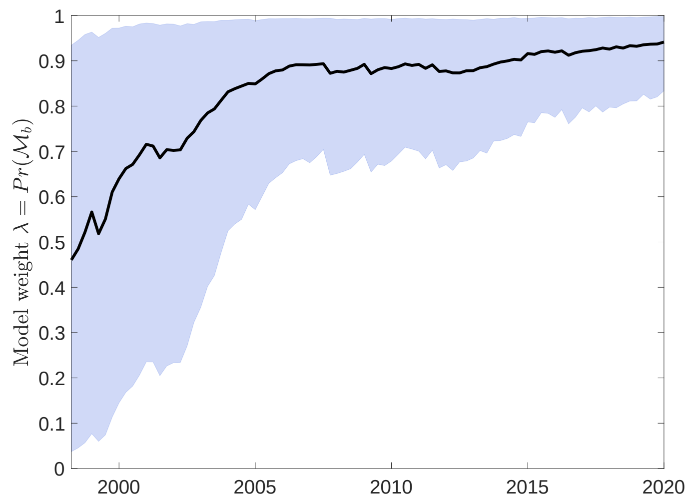
Robustness Exercises: Baseline Model

		$\mathcal{M}_b$	$\mathcal{M}_s$		
Parameters	Description	Infl. Exp. Data	DE	Gabaix	BTHANK
ρ_g	Persistence	0.88	0.87	0.90	0.91
	gov. spending shock	[0.83 0.92]	[0.83 0.91]	[0.86 0.95]	[0.86 0.96]
ρ_ν	Persistence	0.17	0.15	0.18	0.12
	markup shock	[0.08 0.27]	[0.06 0.24]	[0.09 0.28]	[0.04 0.20]
ρ_z	Persistence	0.55	0.47	0.41	0.48
	technology. growth shock	[0.29 0.78]	[0.25 0.69]	[0.19 0.65]	[0.16 0.78]
$100 \times \sigma_g$	Std dev.	0.94	0.95	0.92	0.92
	gov. spending shock	[0.83 1.06]	[0.82 1.07]	[0.80 1.03]	[0.81 1.03]
$100 \times \sigma_\nu$	Std dev.	0.34	0.34	0.39	0.41
	markup shock	[0.29 0.39]	[0.26 0.41]	[0.33 0.45]	[0.35 0.47]
$100 \times \sigma_z$	Std dev.	0.82	0.42	0.78	0.77
	technology growth shock	[0.42 1.26]	[0.25 0.59]	[0.55 1.00]	[0.41 1.12]
$100 \times \sigma_\zeta$	Std dev.	0.37	-	-	-
	sunspot shock	[0.32 0.41]	-	-	-
$\rho(\epsilon_z, \epsilon_\zeta)$	Corr. sunspot and	-0.28	-	-	-
	technlogy growth shocks	[-0.35 -0.21]	-	-	-
$\rho(\epsilon_\nu, \epsilon_\zeta)$	Corr. sunspot and	0.95	-	-	-
	markup shocks	[0.92 0.97]	-	-	-
$\rho(\epsilon_g, \epsilon_\zeta)$	Corr. sunspot and	0.01	-	-	-
	gov. spending shocks	[-0.02 0.05]	-	-	-
$\log\left[p\left(Y^T\right)\right]$	Log-data density	-429.95	-474.58	-451.96	-460.14

Notes: The estimation sample is 1998:Q1 - 2012:Q4. We use $Y^T = [y_1, \ldots, y_T]$ to denote all the available data in our sample. For each model we report posterior means and 90% highest posterior density intervals in square brackets. All posterior statistics are based based on the last 50,000 draws from a RWMH algorithm, after discarding the first 50,000 draws.

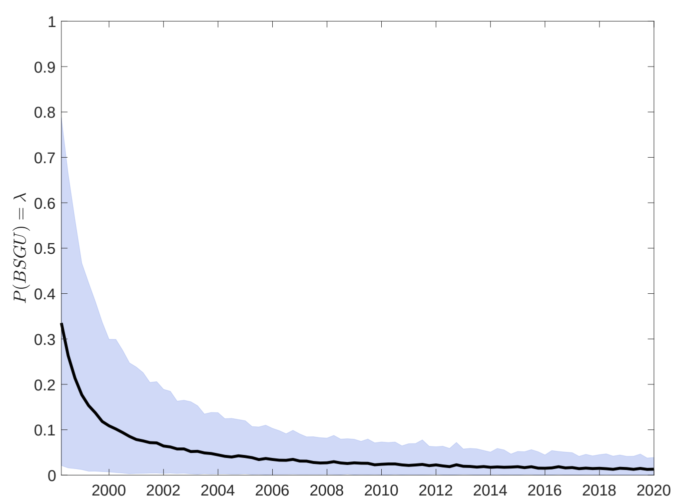
C.7. Prediction Pools: Additional Results

Figure C.2: *Stylized Model:*
Expectations-trap (w/Infl Exp Data) vs Secular Stagnation



Notes: The solid black line is the posterior mean of λ estimated recursively over the period 1998:Q1-2019:Q4 in the stylized model of Section II. The shaded areas correspond to the 90 percent credible set of the posterior distribution. The predictive density is constructed using three observable series: $\Delta y_t^o, \Delta c_t^o, \pi_t^o$.

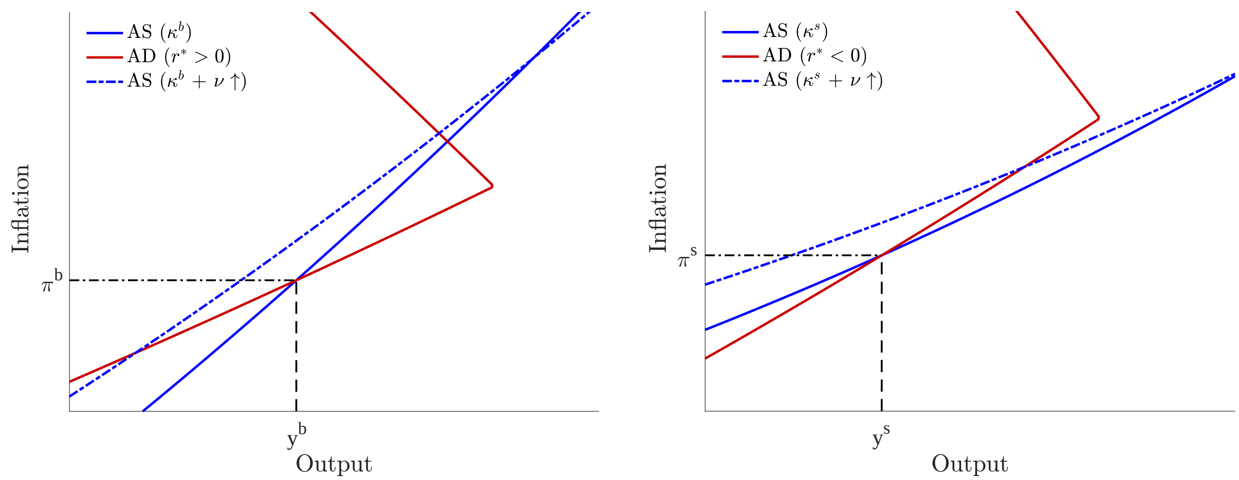
Figure C.3: *Medium Scale Model:*
Expectations-trap (w/o Markup Shocks) vs Secular Stagnation



Notes: The solid black line is the posterior mean of λ estimated recursively over the period 1998:Q1-2019:Q4 in the medium scale model of Section IV. The shaded areas correspond to the 90 percent credible set of the posterior distribution. For the expectations-trap model we set the posterior draws of the parameters $\rho(\epsilon_p, \zeta) = 0$ and $\rho(\epsilon_w, \zeta) = 0$.

Figure D.1: *Permanent increase in markups ν*

(a) *Expectations-Driven Trap* **(b)** *Secular Stagnation*



D. Comparative Statics in the Calibrated Baseline Model

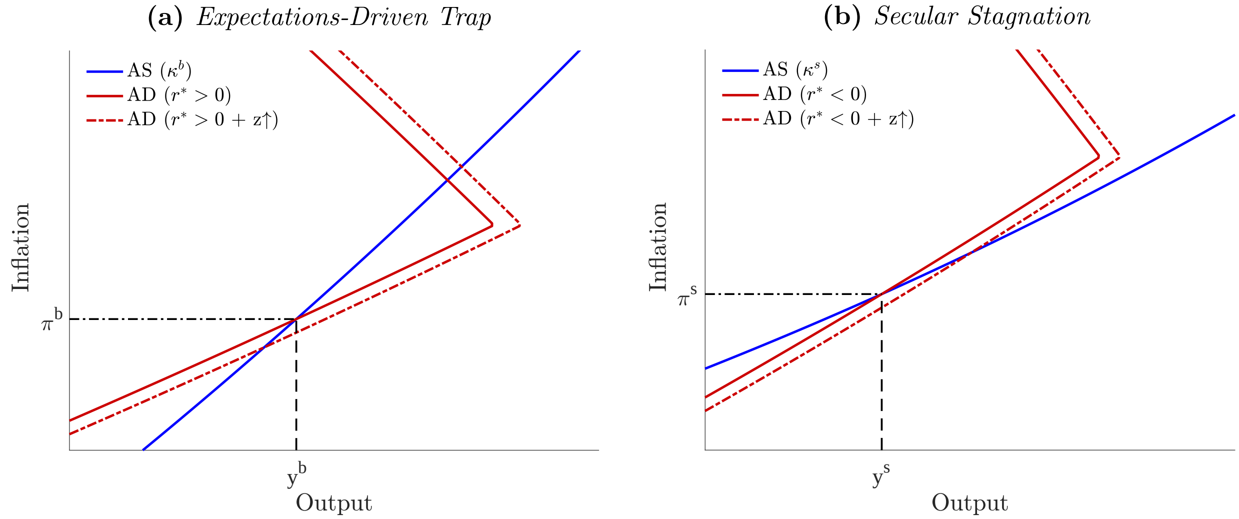
The BSGU and the secular stagnation hypotheses have contrasting implications for shocks and policy. These differences stem from the local determinacy property of these steady states, which translate into differences in slopes of aggregate supply and aggregate demand in our model. We present comparative statics in the calibrated baseline model presented in Section I.

Because of local determinacy of the secular stagnation steady state, the comparative static experiment is well-defined without the need for additional assumptions. With the BSGU steady state, we assume that inflation expectations do not change drastically to push the economy to the full-employment steady state in response to the experiment.

In Figure D.1, solid lines plot the steady-state AD-AS representation of the quantitative model under two parametrizations. Annualized inflation deviation relative to the central bank target inflation is on the vertical axes and output gap relative to target-steady state output (in percents) is on the horizontal axes. The coordinates (y^b, π^b) and (y^s, π^s) denote the expectations-driven and fundamentals-driven liquidity trap steady states respectively. The left panel plots AD-AS curves when prices are relatively flexible (κ^b) and the natural rate of interest is positive. The AD-AS intersection depicted at (y^b, π^b) is locally indeterminate, features zero nominal interest rates and output is permanently below potential. At this intersection, the AS curve is steeper than the AD curve. In the right panel, we plot the AS curve with relatively rigid prices (κ^s), and the AD curve with negative natural interest rate, $r^* < 0$. AD intersects AS at the secular stagnation steady state at the coordinate (y^s, π^s).

An upward shift in aggregate supply curve in Figure D.1, denoted with dashed blue line, induced by permanent increase in steady state markups, translates into higher output under secular stagnation and lower output under BSGU. Under secular stagnation, the natural interest rate is too low for the central bank to stabilize the economy. An increase in markups through inflationary pressures helps lower real interest rate, thus reducing the real interest rate gap and expand output. Under BSGU, the problem is of pessimism about inflation expectations. If agents remain pessimistic about

Figure D.2: *Permanent increase in TFP growth rate z*



inflation undershooting its target, an increase in markups is further contractionary since the resource inefficiencies associated with increased markups dominate the increase in output demand due to higher prices (see also Mertens and Ravn, 2014).

An outward shift in aggregate demand in Figure D.2, denoted with dashed red line, induced by permanent increase in steady state TFP growth, translates to higher output under secular stagnation but lower output under BSGU. Higher TFP growth signals higher income for households and leads to increased consumption demand. This increased impatience translates into higher output under secular stagnation. Under BSGU, in contrast, the increased TFP growth translates into higher reduction in prices by firms, which dominates the increased demand by households. As a result, there is lower output and inflation under BSGU.

Similarly, a neo-Fisherian exit policy of raising interest rates at the ZLB is contractionary under secular stagnation as it increases the real interest rate gap from natural rate, but it is expansionary at the BSGU steady state equilibrium (Schmitt-Grohé and Uribe, 2017). Furthermore, an increase in government expenditure (financed by lumpsum taxes) or a permanent reduction in short-term interest rates below the ZLB has inflationary effects under secular stagnation but deflationary effects under BSGU.[1]

These disparate policy implications raise the question whether it is possible to distinguish these two different kinds of liquidity traps in the data.

[1] We model the neo-Fisherian policy as a permanent change in the intercept of the Taylor rule, a: $R^{new} = \max\{1 + a, a + R^* \left(\frac{\Pi}{\Pi^*}\right)^{\phi_\pi}\} = a + R$. where a is increased to a positive number from zero. This policy simultaneously increases the lower bound on nominal interest rate and thus does not have any effect on the placement of the kink in the aggregate demand curve. Given the inflation rate, an increase in a lowers output demanded. At the secular stagnation steady state, this induces deflationary pressures that increases the real interest rate gap and causes a further drop in output. In contrast, during a BSGU trap, an increase in nominal interest rate anchors agents' expectations to higher levels of inflation, thus obtaining the neo-Fisherian results (Schmitt-Grohé and Uribe, 2017). The effects of increased government spending on output are somewhat ambiguous because of elastic labor supply that also causes changes in the aggregate supply curve.

E. Steady State Analysis with Linear Phillips Curve

With the help of some simplifying assumptions in the derivation of the new Keynesian Phillips curve, we can analytically characterize the steady states in the baseline model. All the ingredients of the model are same as studied in Section I except we assume that the cost of price adjustment takes the following functional form:

$$\Phi_s \equiv \frac{\phi}{2} \frac{\left(\frac{P_t(j)}{P_{t-1}} - \Pi^*\right)^2}{\Pi_t} Y_t$$

This adjustment cost function combines the insights of Bilbiie (2021) to make the Phillips curve static, along with the insights of Bhattarai, Eggertsson and Gafarov (2022) which deliver an aggregate Phillips curve that is linear. Under the assumption of infinitely elastic labor supply ($\eta \to \infty$), and adjustment cost function of the form Φ_s, we get the following static aggregate Phillips curve relationship between aggregate inflation rate Π_t and stationary consumption $\tilde{C}_t$:[2]

$$\Pi_t = \frac{1-\nu}{\phi\nu_t}\tilde{C}_t + \left(\Pi^* - \frac{1-\nu_t}{\phi\nu_t}\right) \tag{E.1}$$

This static Phillips curve is the Rotemberg costs equivalent of the analytical Phillips curve derived by Bhattarai et al. (2022). Other than that, the model is same as the baseline model.

We assume there is no government spending in the steady state for now (i.e. $g = 1$). From equations 1, 3, 4, and E.1 we can represent the steady-state equilibrium with an aggregate demand block and an aggregate supply block. We describe each of these blocks next.

Aggregate Demand (AD) is a relation between output and inflation and is derived by combining the Euler equation and the Taylor rule. The AD curve is given by

$$\tilde{Y}_{AD} = \frac{1}{\delta}\begin{cases} 1 - \beta r \left(\frac{\Pi}{\Pi^*}\right)^{\phi_\pi - 1}, & \text{if R } > 1, \\ 1 - \frac{\beta}{\Pi} \quad , & \text{if R } = 1. \end{cases} \tag{E.2}$$

When the ZLB is not binding, the AD curve has a strictly negative slope; and it becomes linear and upward sloping when the ZLB constrains the nominal interest rate. Thus, the kink in the aggregate demand curve occurs at the inflation rate where the ZLB constrains monetary policy: $\Pi_{kink} = \left(\frac{1}{r\Pi^*}\right)^{\frac{1}{\phi_\pi}} \Pi^*$. For the natural interest rate to be positive $r > 1$, the premium on government bonds must be low enough i.e. $\delta < 1 - \beta$.

Aggregate Supply (AS) is given by $\Pi = \kappa\tilde{Y} + (\Pi^* - \kappa)$ in the steady state, where $\kappa \equiv \frac{1-\nu}{\phi\nu}$. When $h = 1$, $\Pi = \Pi^*$. In this linear aggregate supply curve, the degree of nominal rigidity κ also determines a lower bound on the inflation $= \Pi^* - \kappa$.

In this two-equation representation, we can characterize different steady-state equilibria. Proposition E.1 shows that a targeted steady state exists as long as the natural interest rate is positive.

[2] Note that ω is set equal to $(1-\nu)$ in the steady state to target $h = 1$.

Proposition E.1. (Targeted Steady State): Let $\Pi^* = 1$, $0 < \delta < 1 - \beta$. There exists a unique positive interest rate steady state with output at potential $\tilde{Y} = 1$, inflation at target $\Pi = 1$ and $R = \frac{1-\delta}{\beta} > 1$. It features output at efficient steady state, and inflation at the policy target. The equilibrium dynamics in this steady state's neighborhood are locally determinate.

Proof. The downward sloping portion of aggregate demand always goes through $Y = 1$ and $\Pi = 1$ when $\Pi^* = 1$. When $\delta < 1 - \beta$, $r > 1$. The kink in the AD curve occurs at inflation rate below 1. There always exists an intersection between the AS and the AD at $Y = 1, \Pi = 1$. To show that there does not exist another steady state at positive interest rates, note the AS curve is linear and upward sloping. For $\Pi > 1$, $Y_{AD} < 1 < Y_{AS}$, and for $\Pi_{kink} \leq \Pi < 1$, $Y_{AD} > 1 > Y_{AS}$. Thus, there does not exist another steady state at positive nominal interest rate.

To prove local determinacy, log-linearize the equilibrium conditions around the unique non-stochastic steady state, and it follows that the Blanchard-Kahn conditions for determinacy are satisfied as $\phi_\pi > 1$. □

A steady state at which the central bank can meet its inflation target is *defined* as the *targeted-inflation steady state*. The presence of a targeted-inflation steady state is contingent on the natural interest rate and the monetary authority's inflation target. With a unitary inflation target, it must be the case that the natural interest rate is non-negative, which is implied by the assumption of $0 < \delta < 1 - \beta$. In Proposition E.2 we show that, a liquidity trap steady state (à la Schmitt-Grohé and Uribe, 2017) may jointly co-exist with the targeted steady state described above. However, with a flat enough Phillips curve, a targeted steady state is the unique steady state in this economy. A high enough nominal rigidity prevents inflation from falling to levels such that self-fulfilling deflationary expectations do not manifest in the steady state.

Proposition E.2. (Deflationary expectations-trap steady state): Let $\Pi^* = 1$, $0 < \delta < 1 - \beta$. For $\kappa > 1 - \beta$ (i.e. $\phi < \frac{1-\nu}{\nu(1-\beta)}$) there exist two steady states:

1. The targeted steady state with output at potential $\tilde{Y} = 1$, inflation at target $\Pi = 1$ and positive nominal interest rate $R = \frac{1-\delta}{\beta} > 1$.

2. (*Deflationary expectations-driven* trap) A unique-ZLB steady state with output below potential $\tilde{Y} < 1$, inflation below target $\Pi < 1$ and zero nominal interest rate $R = 1$. The local dynamics in a neighborhood around this steady state are locally indeterminate.

When prices are rigid enough, i.e., $\kappa < 1 - \beta$, there exists a unique steady state, and it is the targeted inflation steady state. When prices are flexible $\phi = 0$ ($\kappa \to \infty$), two steady states exist. A unique deflationary steady state with zero nominal interest rates and a unique targeted inflation steady state.

Proof. When $\delta < 1 - \beta$, $R^* > 1$. The kink in the AD occurs at inflation rate below 1. Following the steps in Proposition E.1, it follows that a targeted steady state with output at potential, inflation at target, and positive nominal interest rate exists. At Π_{kink}, $Y_{AD} > 1 > Y_{AS}$. When $\Pi = 1 - \kappa$, $Y_{AS} = 0 > Y_{AD}$. Given that AS is linear and AD is strictly convex, there is a unique intersection at positive unemployment. At this second steady state, AS intersects AD from above. In the model, the relative steepness of AS curve is equivalent to locally indeterminate dynamics. □

We define the *deflationary expectations-driven* trap as the steady state with a positive natural interest rate, negative output gap, and deflation and in whose neighborhood the equilibrium dynamics are *locally indeterminate*. Pessimistic inflationary expectations can push the economy to this steady state without any change in fundamentals.

We now consider the case where adverse fundamentals can push the economy to a permanent liquidity trap. If agents are sufficiently patient $\delta > \frac{1}{\beta}$, i.e., the natural rate of interest is negative, and the ZLB constrains monetary policy. In that case, the nominal interest rate is permanently zero while there is below-potential output and deflation in the economy. We characterize this possibility in Proposition E.3.

Proposition E.3. (Secular Stagnation): Let $\Pi^* = 1$, $\delta > 1 - \beta$ and $\kappa < 1 - \beta$. There exists a unique steady state with output below potential $\tilde{Y} < 1$, inflation below target $\Pi < 1$ and zero nominal interest rate $R = 1$. It features output below the targeted steady state and deflation, caused by a permanently negative natural interest rate. The equilibrium dynamics in this steady state's neighborhood are locally determinate.

Proof. When $\delta > 1-\beta$, $R^* < 1$. Thus, the kink in the AD occurs at inflation rate above 1. AS is linear and AD is strictlyconvex. At Π_{kink}, $Y_{AD} < 1 < Y_{AS}$. The last inequality requires the assumption that $\kappa < 1 - \beta$. Thus, the AS and AD must intersect at positive unemployment. With log-linearized equilibrium conditions, it can be seen that the system is locally determinate. In this case, the local determinacy condition is equivalent to the AD curve being steeper than the AS curve at the stagnation steady state.

□

We formally *define* the *secular stagnation steady state* as the steady state featuring negative output gap, zero nominal interest rate on short-term government bonds and exhibiting *locally determinate* equilibrium dynamics in its neighborhood. This local determinacy property is the main difference between the secular stagnation narrative and the expectations-driven narrative.

Note that the secular stagnation steady state exists in this model because of sufficient discounting in the modified Euler equation. Unlike the traditional new Keynesian model, an arbitrarily long ZLB episode driven by a negative natural rate can exist in equilibrium. In log-linearized new Keynesian models without discounting, *deflationary black holes* emerge as the duration of the temporary liquidity trap is increased, with inflation and output tending to negative infinity (Eggertsson, 2011). The solution remains bounded in our setup as the duration of ZLB is increased.

F. Medium-Scale DSGE Model

Here we provide a detailed derivation of the model in Section IV. The exposition follows Gust, Herbst, López-Salido and Smith (2017). There are five agents in the economy: (i) monopolistically competitive intermediate goods firms (ii) a perfectly competitive firm that aggregates the differentiated varieties from intermediate producers; (iii) a perfectly competitive employment agency that bundles households' labor services (iv) a continuum of households that make optimize consumption, investment, capital utilization, and supply differentiated labor services to a labor agency; (v) the government that sets fiscal and monetary policy.

F.1. Model Description

Intermediate and final goods firms

Intermediate goods producers sell the intermediate varieties Y_{jt} to the final good firms that produce the final composite good: $Y_t = \left[\int_0^1 Y_{jt}^{1-\nu_{p,t}} dj\right]^{\frac{1}{1-\nu_{p,t}}}$, where $1/\nu_{p,t} > 1$ is a time-varying price-markup $\lambda_t^p = \frac{1}{1-\nu_{p,t}}$. The demand for intermediate good j has the iso-elastic form $Y_{jt} = \left(\frac{P_{jt}}{P_t}\right)^{-1/\nu_{p,t}} Y_t$ where P_{jt} is the price of variety j and P_t is the aggregate price index. Each intermediate good j is produced by a price-setting monopolist using the following technology: $Y_{jt} = (Z_t L_{jt})^{1-\alpha} \ K_{jt}^{\alpha}$. Where L_{jt} is the labor input, K_{jt} is physical capital, and Z_t denotes a non-stationary TFP process. The growth rate of Z_t denoted by $G_{Z,t}$ follows an AR(1) process with persistence ρ_z and iid shocks $\epsilon_{z,t} \sim iid\ N(0, \sigma_z^2)$ that causes deviations of the TFP growth from balanced growth rate G_Z.

Firms choose inputs to minimize total cost each period. Cost minimization implies that the capital-labor ratio at the firm level is independent of firm-specific variables $\frac{K_{jt}}{L_{jt}} = \frac{K_t}{L_t} = \frac{\alpha}{1-\alpha}\frac{W_t}{R_t^k}$. As a result, nominal marginal costs are $P_t mc_t = \frac{1}{Z_t^{1-\alpha}}\left(\frac{R_t^k}{\alpha}\right)^{\alpha}\left(\frac{W_t}{1-\alpha}\right)^{1-\alpha}$. To set the price P_{jt}, intermediate firm j pays a quadratic adjustment cost in units of final good $\frac{\phi_p}{2}\left(\frac{P_{jt}}{\tilde{\Pi}_{t-1}P_{jt-1}} - 1\right)^2 P_t Y_t$. Where $\phi_p \geq 0$ is the parameter that scales the cost of price changes and price adjustments are indexed to $\tilde{\Pi}_{t-1} = \bar{\Pi}^{1-\iota_p}\Pi_{t-1}^{\iota_p}$, where ι_p governs indexation between previous period inflation rate Π_{t-1} and steady-state inflation rate $\bar{\Pi}$. Firm's per period profits are given by: $\Gamma_t \equiv P_{jt}Y_{jt} - P_t mc_t N_{jt} - \frac{\phi_p}{2}\left(\frac{Pjt}{\tilde{\Pi}_{t-1}Pjt-1} - 1\right)^2 P_t Y_t$. And the firm's profit maximization problem is

$$\max_{Pjt}\left\{\Gamma_t + E_t\sum_{s=1}^{\infty} Q_{t,t+s}\Gamma_{t+s}\right\} \tag{F.1}$$

where $Q_{t,t+s}$ is the nominal stochastic discount factor of the household.

Households

A continuum of households, indexed by $i \in [0,1]$, supply differentiated labor services L_{it} to a perfectly competitive labor agency. The agency combines labor services into a homogeneous labor composite L_t according to $L_t = \left[\int_0^1 L_{i,t}^{1-\nu_{w,t}} di\right]^{\frac{1}{1-\nu_{w,t}}}$, where $1/\nu_{w,t} > 1$ is the elasticity of demand that determines the time-varying wage-markup $\lambda_t^w = \frac{1}{1-\nu_{w,t}}$. The demand for labor inputs of type j is $L_{j,t} = \left(\frac{W_{j,t}}{W_t}\right)^{-1/\nu_{w,t}} L_t$, where W_{it} is the wage set by the union on behalf of the workers, and W_t is the aggregate nominal wage. At time t, the household-i chooses consumption C_{it}, risk-free nominal bonds B_t, investment I_t and capital utilization u_t to maximize the utility function, with external habits in consumption:

$$\mathbb{E}_t \Sigma_{s=t}^{\infty} \beta^{s-t} \left[\log(C_{j,s} - hC_{j,s-1}) - \frac{\omega}{1+1/\eta} L_{j,s}^{1+1/\eta} + \delta_t \frac{B_{t+1}}{Z_t P_t} - \psi_{j,s}^w \right], \tag{F.2}$$

where h is the degree of habit formation on internal habits over individual consumption. $\eta > 0$ is the Frisch elasticity of labor supply, $\omega > 0$ is a parameter that pins down the steady-state level of hours, exogenous parameter δ_t regulates the utility from bonds,[3] and the discount factor β satisfies $0 < \beta < 1$. The utility loss, $\psi_{j,t}^w$, due to adjustments in the nominal wage takes the form $\psi_{j,t}^w = \frac{\phi_w}{2} \left[\frac{W_{jt}}{\tilde{\Pi}_{t-1}^w W_{jt-1}} \right]^2$. Where $\phi_w \geq 0$ is a parameter, wage contracts are indexed to productivity and price inflation. We assume $\tilde{\Pi}_{t-1}^w = G_Z \bar{\Pi}^{1-\iota_w} \left(\exp(\epsilon_{Z,t}) \Pi_{t-1} \right)^{\iota_w}$ with $0 \leq \iota_w < 1$. We assume perfect consumption risk sharing across the households. As a result, the household's budget constraint in period t is given by

$$P_t C_{i,t} + P_t I_{i,t} + \frac{B_{i,t+1}}{(1+i_t)} = B_{i,t} + B_{i,t}^S + W_t L_{i,t} + \Gamma_t + T_t + R_t^K u_{i,t} K_t^u - P_t a(u_{i,t}) K_{i,t}^u\,, \tag{F.3}$$

where $I_{i,t}$ is investment, $B_{i,t}^S$ is the net cash flow from household i's portfolio of state-contingent securities. Households own an equal share of all firms and thus receive Γ_t dividends from profits. Finally, each household receives a lump-sum government transfer T_t. Since households own capital, they choose the utilization rate. The amount of effective capital that the households rent to the firms at the nominal rate R_t^K is $K_{i,t} = u_t K_{i,t}^u$. The unit nominal cost of capital utilization is $P_t a(u_{i,t})$. As in the literature (Smets and Wouters, 2007) we assume $a(1) = 0$, and the elasticity of utilization costs is parameterized by $a''(1) > 0$. Following Christiano, Eichenbaum and Evans (2005), we assume investment adjustment costs in the production of capital. The law of motion for capital is as follows:

$$K_{i,t+1}^u = \mu_t \left[1 - S\left(\frac{I_{i,t}}{G_Z I_{i,t-1}} \right) \right] I_{i,t} + (1-\delta_k) K_{i,t}^u \tag{F.4}$$

Where G_Z is the steady state growth rate of Z_t, δ_k is the depreciation rate of capital, and μ_t is an exogenous disturbance to the marginal efficiency of investment. We assume the presence of investment adjustment costs that satisfy $S(1) = S'(1) = 0$, and the elasticity of adjustment costs to investment changes is given by $S^{''}(1) > 0$.

Fiscal and Monetary Policy

We assume the government balances the budget every period $P_t T_t = P_t G_t$ and G_t is the government spending, which is determined exogenously as a fraction of GDP $G_t = \left(1 - \frac{1}{g_t}\right) Y_t$, where g_t is an exogenous shock to government spending.

[3]We assume that δ_t evolves as an AR(1) process, labeled as η_t later in the model.

The central bank sets the nominal interest rate i_t, following an inertial rule that responds to deviations of inflation from a constant target $\bar{\Pi}$, and output growth relative to the economy's long-run growth rate.

$$\frac{1+i_t}{1+i_{ss}} = \max\left(\frac{1}{1+i_{ss}}, \left(\frac{1+i_{t-1}}{1+i_{ss}}\right)^{\rho_R}\left[\left(\frac{\Pi_t}{\bar{\Pi}}\right)^{\phi_\pi}\left(\frac{Y_t}{G_Z Y_{t-1}}\right)^{\phi_{dy}}\right]^{1-\rho_R}\right), \tag{F.5}$$

where i_{ss} is the steady state nominal interest rate, Π_t is the gross inflation rate, ρ_R determines the degree of inertia in interest rate changes. The first term inside the max operator captures the effective lower bound on the nominal interest rate. Throughout our analysis, we will assume that such a lower bound will be binding.

Market clearing

We focus on a symmetric equilibrium that satisfies $K_t = \int_0^1 K_{i,t}di$, $N_t = \int_0^1 L_{i,t}di = \int_0^1 L_{j,t}dj$. In this symmetric equilibrium, the market clearing for the final good requires

$$Y_t = C_t + I_t + a(u_t)K_t^u + G_t + \frac{\phi_p}{2}\left[\frac{\Pi_t}{\tilde{\Pi}_{t-1}} - 1\right]^2 Y_t \tag{F.6}$$

F.2. Equilibrium Conditions

We present the model's equilibrium conditions as stationary variables. Let Z_t be the non-stationary level of TFP at time t. We normalize the following variables :

$$y_t = Y_t/Z_t\,,$$

$$c_t = C_t/Z_t\,,$$

$$k_t = K_t/Z_t\,,$$

$$k_t^u = K_t^u/Z_{t-1}\,,$$

$$\mathbb{I}_t = I_t/Z_t\,,$$

$$w_t = W_t/(Z_t P_t)\,,$$

$$r_t^k = R_t^k/P_t\,,$$

$$\lambda_t = \Lambda_t Z_t\,,$$

Definition 1 (Normalized equilibrium)**.** 17 endogenous variables $\{\lambda_t,\, i_t,\, c_t,\, y_t,\, \Pi_t,\, mc_t,\, \tilde{\Pi}_{t-1},\, \Pi_t^w,\, \tilde{\Pi}_{t-1}^w,\, w_t,\, L_t,\, k_{t+1}^u,\, r_t^K,\, \mathbb{I}_t,\, q_t,\, u_t,\, k_t\}$, 6 endogenous shock processes $\{z_t,\, g_t,\, \eta_t,\, \mu_t,\, \nu_{p,t},\, \nu_{w,t}\}$, 6 exogenous shocks $\{\epsilon_{z,t}, \epsilon_{g,t}, \epsilon_{\eta,t}, \epsilon_{\mu,t}, \epsilon_{\nu_p,t}, \epsilon_{\nu_w,t}\}$ given initial values of k_{t-1}^u.

Consumption Euler equation

$$\lambda_t = \beta(1+i_t)\mathbb{E}_t\left[\frac{\lambda_{t+1}}{G_{Z,t+1}}\frac{1}{\Pi_{t+1}}\right] + \delta_t\,, \tag{F.7}$$

$$\lambda_t = \frac{1}{c_t - \frac{hc_{t-1}}{G_{Z,t}}} - h\beta\mathbb{E}_t \frac{1}{G_{Z,t+1}\left[c_{t+1} - \frac{hc_t}{G_{Z,t+1}}\right]}\,, \tag{F.8}$$

Price-setting

$$(1-\nu_{p,t}) - mc_t + \nu_{p,t}\phi_p\left(\frac{\Pi_t}{\tilde{\Pi}_{t-1}} - 1\right)\frac{\Pi_t}{\tilde{\Pi}_{t-1}} - \nu_{p,t}\phi_p\beta E_t\frac{\lambda_{t+1}}{\lambda_t}\left(\frac{\Pi_{t+1}}{\tilde{\Pi}_t} - 1\right)\frac{\Pi_{t+1}}{\tilde{\Pi}_t}\frac{y_{t+1}}{y_t} = 0 \tag{F.9}$$

$$\tilde{\Pi}_{t-1} = \bar{\Pi}^{1-\iota_p}\Pi_{t-1}^{\iota_p} \tag{F.10}$$

Wage-setting

$$\nu_{w,t}\phi_w\left[\frac{\Pi_t^w}{\tilde{\Pi}_{t-1}^w} - 1\right]\frac{\Pi_t^w}{\tilde{\Pi}_{t-1}^w} = \nu_{w,t}\phi_w\beta\mathbb{E}_t\left[\frac{\Pi_{t+1}^w}{\tilde{\Pi}_t^w} - 1\right]\frac{\Pi_{t+1}^w}{\tilde{\Pi}_t^w} + L_t\lambda_t\left[\omega\frac{L_t^{\frac{1}{\eta}}}{\lambda_t} - (1-\nu_{w,t})w_t\right] \tag{F.11}$$

$$\tilde{\Pi}_{t-1}^w = G_Z\bar{\Pi}^{1-\iota_w}\left(\exp(\hat{G}_{z,t})\Pi_{t-1}\right)^{\iota_w} \tag{F.12}$$

$$\Pi_{W,t} = \frac{w_t}{w_{t-1}}\Pi_t G_{Z,t}\,, \tag{F.13}$$

Capital investment

$$k_{t+1}^u = \mu_t\left[1 - S\left(\frac{\mathbb{I}_t}{\mathbb{I}_{t-1}}\frac{G_{Z,t}}{G_Z}\right)\right]\mathbb{I}_t + (1-\delta_k)\frac{k_t^u}{G_{Z,t}}\,, \tag{F.14}$$

$$q_t = \beta\mathbb{E}_t\left[\frac{\lambda_{t+1}}{\lambda_t G_{Z,t+1}}\left(r_{t+1}^K u_{t+1} - a(u_{t+1}) + q_{t+1}(1-\delta_k)\right)\right]\,, \tag{F.15}$$

$$\begin{aligned} &q_t\mu_t\left[1 - S\left(\frac{\mathbb{I}_t}{\mathbb{I}_{t-1}}\frac{G_{Z,t}}{G_Z}\right) - S'\left(\frac{\mathbb{I}_t}{\mathbb{I}_{t-1}}\frac{G_{Z,t}}{G_Z}\right)\frac{\mathbb{I}_t}{\mathbb{I}_{t-1}}\frac{G_{Z,t}}{G_Z}\right] \\ &+\beta\mathbb{E}_t\left[\mu_{t+1}\frac{\lambda_{t+1}}{\lambda_t}q_{t+1}\frac{G_{Z,t+1}}{G_Z}\left(\frac{\mathbb{I}_{t+1}}{\mathbb{I}_t}\right)^2 S'\left(\frac{\mathbb{I}_{t+1}}{\mathbb{I}_t}\frac{G_{Z,t+1}}{G_Z}\right)\right] = 1 \end{aligned} \tag{F.16}$$

Capital utilization rate

$$k_t = u_t\frac{k_t^u}{G_{Z,t}}\,, \tag{F.17}$$

$$r_t^K = a'(u_t)\,, \tag{F.18}$$

Production technologies

$$y_t = k_t^\alpha L_t^{1-\alpha}\,, \tag{F.19}$$

$$r_t^k = \alpha mc_t\frac{y_t}{k_t}\,, \tag{F.20}$$

$$w_t = (1-\alpha)mc_t\frac{y_t}{L_t}\,, \tag{F.21}$$

Government

$$\frac{1+i_t}{1+i_{ss}} = \max\left(\frac{1}{1+i_{ss}}, \left(\frac{1+i_{t-1}}{1+i_{ss}}\right)^{\rho_R}\left[\left(\frac{\Pi_t}{\bar{\Pi}}\right)^{\phi_\pi}\left(\frac{y_t G_{Z,t}}{y_{t-1}G_Z}\right)^{\phi_{dy}}\right]^{1-\rho_R}\exp(\epsilon_{mp,t})\right), \tag{F.22}$$

Market clearing

$$y_t = c_t + \mathbb{I}_t + a(u_t)\frac{k_t^u}{G_{Z,t}} + \left(1-\frac{1}{g_t}\right)y_t\,, \tag{F.23}$$

Law of Motion of Shocks The six structural shocks driving the model economy are assumed to follow first-order auto-regressive processes of the form $\log(x_t) = (1-\rho_x)\log(x) + \rho_x log(x_{t-1}) + \sigma_x\epsilon_{x,t}$, with $\epsilon_{x,t} \sim N(0,1)$, and x denoting steady-state values, for $x = G_Z, g, \eta, \mu, \nu_p, \nu_w$.

F.3. Parameters and Model Calibration

Table F.1 details the parameters for the medium-scale DSGE model of section IV that are fixed before estimation in both the secular stagnation and expectations-trap models. We source most of the parameters from the estimated DSGE model for Japan in Hirose (2020). Our medium-scale model shares many features with the estimated model in Hirose (2020). We rely on the existing estimates of certain structural parameters to focus our empirical assessment on the equilibrium dynamics implied by our alternative stagnation hypothesis. The average growth rate of productivity, G_z, is set to match the average growth rate of GDP from 1998:Q1-2012:Q4. The steady-state value of the government spending shock, g, is set to match the share of the autonomous spending-to-GDP ratio during the same period. In the medium-scale model, autonomous spending includes nominal government spending and net exports.

Lastly, Table F.1 also shows the value of the parameters calibrated for each model independently. We also show the respective calibration targets. As discussed in Section I, the marginal utility of bond holdings, δ, pins the natural rate of interest in our model. The parameter ψ_p, which controls the cost of adjusting prices, is set such that the prices are twice as flexible in the secular stagnation model compared with the expectations-trap model. Given the natural rate and the cost of price adjustment, the parameter ψ_w, which controls the cost of adjusting wages, is endogenously calibrated to match a steady inflation of -1.06 observed in our sample. To keep both secular stagnation and expectation traps on equal footing, we set the disutility of hour worked ω such that in the steady state with a binding zero lower bound, hours worked are below steady state hours worked under full employment.

Table F.1: *Structural Parameters: Medium Scale Model*

Common Parameters					
β	α	γ	$1/(1-\nu_p)$	$1/(1-\nu_w)$	γ_p
Discount factor	Capital share	Habit persistence	Price markup	Wage markup	Price indexation
0.942	0.37	0.36	1.2	1.2	0.225
γ_w	$a^{''}(1)$	$S^{''}(1)$	δ_k	g	G_z
Wage indexation	Utilization elasticity	Inv. Adjustment cost	Capital depreciation	Autonomous spending	TFP growth rate
0.295	2.246	5.16	0.015	1.33	0.26

Model Specific Parameters					
Expectations-trap				Secular stagnation	
δ	ϕ_p			δ	ϕ_p
Mg. utility bonds	Price adj. cost			Mg. utility bonds	Price adj. cost
0.0405	800			0.0415	1800
ϕ_w	ω			ϕ_w	ω
Wage adj. cost	Labor disutility			Wage adj. cost	Labor disutility
653.8	0.62			688.2	0.59

Notes: The parameter β is calibrated to obtain a positive natural rate in the presence of positive bond premium—see Section I.D. The parameters $\alpha, \gamma, \gamma_p, \gamma_w, a^{''}(1), S^{''}(1), \delta_k$ are taken from Hirose (2020, Table 2). The parameters ν_p and ν_w target 20 percent price and wage markups in the steady state. The parameters g, and G_z are set to match the average GDP share of autonomous spending $(G + X - M)/Y$ and average per-capita GDP growth from 1998Q1 to 2012Q4, respectively. The model-specific parameters are calibrated as discussed in section 5.1.

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